Exercise Sheet 9 due 8 January 2015

1. *spherical harmonics*

   i. Using the spherical harmonics for $l = 1$ derived in the lecture, calculate $|Y(\vartheta,\varphi)|^2$ for $Y(\vartheta,\varphi) = \frac{1}{\sqrt{3}}\,(Y_{1,-1}(\vartheta,\varphi) + Y_{1,0}(\vartheta,\varphi) + Y_{1,1}(\vartheta,\varphi))$.

   ii. Write the spherical harmonics for $l = 1$ in Cartesian coordinates, i.e., replace $\sin(\vartheta)\cos(\varphi)$ by x, $\sin(\vartheta)\sin(\varphi)$ by y, and $\cos(\vartheta)$ by z.

   iii. Write the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in the space of the spherical harmonics with $l = 1$, i.e., the 3×3 matrices $\langle l = 1, m|\hat{L}_i|l = 1, m'\rangle$. Do the same for $\hat{L}_\pm$ and $\vec{L}^2$.

   iv. Using the ladder operators for orbital angular momentum, derive the spherical harmonics for $l = 2$:

$$
\begin{aligned}
Y_{2,-2}(\vartheta,\varphi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\vartheta\, e^{-2i\varphi} \\
Y_{2,-1}(\vartheta,\varphi) &= \sqrt{\frac{15}{8\pi}}\,\sin\vartheta\cos\vartheta\, e^{-i\varphi} \\
Y_{2,0}(\vartheta,\varphi) &= \sqrt{\frac{5}{16\pi}}\,(3\cos^2\vartheta - 1) \\
Y_{2,1}(\vartheta,\varphi) &= -\sqrt{\frac{15}{8\pi}}\,\sin\vartheta\cos\vartheta\, e^{i\varphi} \\
Y_{2,2}(\vartheta,\varphi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\vartheta\, e^{2i\varphi}
\end{aligned}
$$

   v. Write the spherical harmonics for $l = 2$ in Cartesian coordinates.

2. *positronium*

Positronium is a hydrogen-like system, where a positron (the anti-particle of an electron with charge $+e$ and mass m_e) takes the role of the proton (hydrogen nucleus). What is the reduced mass to be used when solving the positronium problem?